Introduction to Geometry (Autumn Tertm 2012) Exercises 3

Section A

1. Show that in Figure 1(i) below, $AP \cdot PC = BP \cdot PD$ (Euclid III.35).

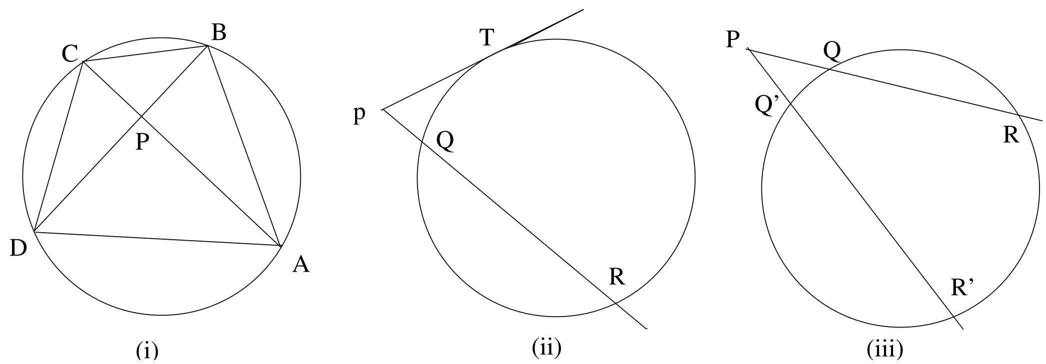


Figure 1

2. What is the angle-sum in a quadrilateral on the sphere, whose edges are arcs of great circles?

3. Show that the three medians of a (plane) triangle are concurrent. (Hint: Ceva's theorem).

4. Euclidean construction of a segment whose length is the product of two given lengths (given a unit of length): segments of lengths $1, a$ and b are positioned as shown. The line RS is drawn parallel to PQ. Show that the segment marked x has length ab.

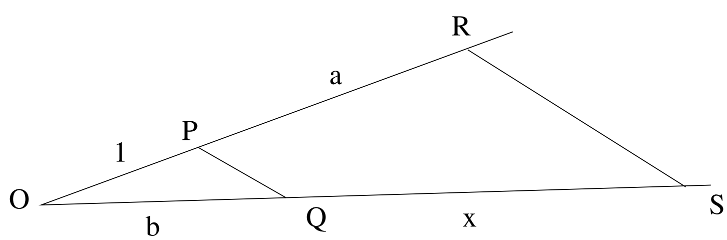


Figure 2

Section B

5. (i) Show that in Figure 1(iii) we have $PQ \cdot PR = PQ' \cdot PR'$.
(ii) Show that in Figure 1(ii) we have $PT^2 = PQ \cdot PR$ (Euclid III.36).

6. The diagram below shows two regular pentagons side by side.

(i) Find the angle $\angle DBC$. (Hint: mark the centre of the right-hand pentagon and use Exercises Sheet 1, Question 2.)

(ii) Show that ABC is a straight line.

(iii) Find the ratio d/s.

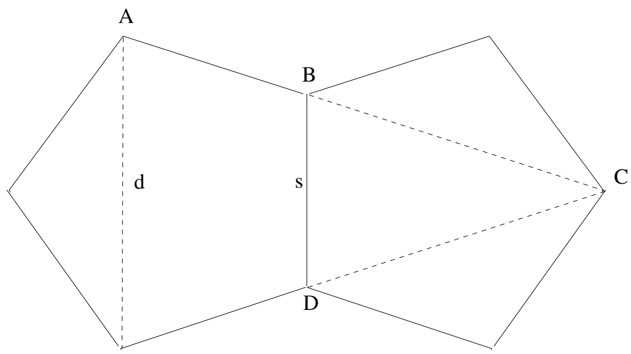


Figure 3

7. The formula $1/f = 1/u + 1/v$ is important in optics, where it describes the relation between the object and image distances u and v from a mirror which has focal length f. It is used in the preparation of optical equipment, to determine one of the variables from knowledge of the other two. To avoid calculation, lens-makers used to use the *nomogram*, a diagram which allows you to find the unknown variable by measurement. The nomogram consists of three axes meeting at O, each making an angle of $\pi/3$ to its neighbour. The procedure is as follows: given two out of u, v and f, mark off the known values on their corresponding axes, draw a line through the marked points, and then find the unknown variable by seeing where this line crosses the remaining axis.

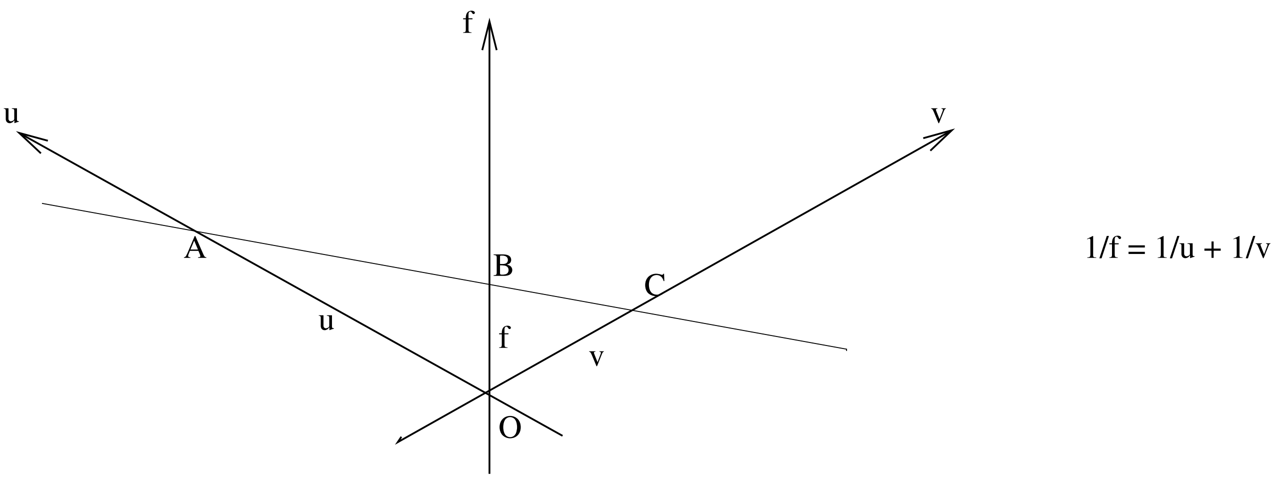


Figure 4

Why does this work? That is, why is $1/f$ equal to $1/u + 1/v$? Hint: draw X on the **f**-axis so that AX is parallel to the **v**-axis. Look first for an equilateral triangle and then for some similar triangles.

Section C

8. Find a construction, given segments of length 1 and a, for a segment whose length is $1/a$.

9. Here is a construction for a segment whose length is the square root of the length of a given segment AB. Continue AB to C so that BC has length 1. Draw the circle with diameter AC, and draw also the line perpendicular to AC through B. If this line meets the circle at X and at Y, then $BX = BY = \sqrt{AB}$. Why?

10. (i) In the diagram below, show that the two rectangles have the same area if and only if the two sloping lines are parallel.

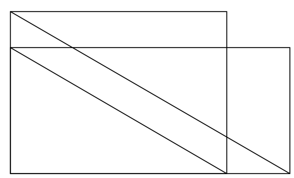
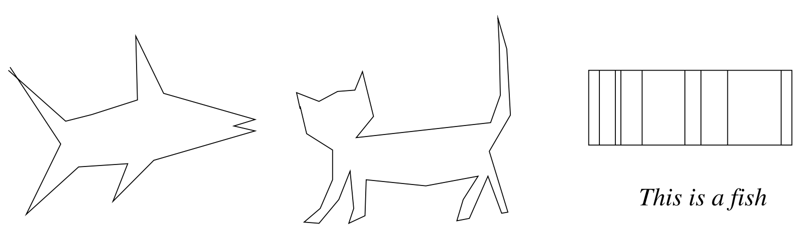


Figure 5

(ii) Deduce that the taller rectangle can be cut up into pieces and re-assembled to form the shorter one.
(iii) Show that a right angled triangle can be cut up into 2 pieces and reassembled as a rectangle. Show that an arbitrary triangle can be cut up into four pieces and reassembled as a rectangle. Proofs here should consist mainly of good diagrams.

11. (i) Exercise 10 gives the first steps in the proof of the Plane Dissection Theorem, which says that given two plane polygons with the same area, it is possible to cut one up and reassemble the pieces to get the other. A crucial step in this proof is to show that the theorem is true for rectangles, and Exercise 10 (i) and (ii) does this, provided the tall thin rectangle is not more than twice as high as the short fat one. If it is more than twice as high, the upper of the two sloping lines goes outside the union of the two rectangles, and the argument breaks down. Nevertheless, a simple preliminary step solves this problem. What is it?
(ii) Prove the Plane Dissection Theorem. First hint: first divide into triangles. Second hint:



Historical Note: At the start of the last century, the famous German mathematician David Hilbert described a list of 20 unsolved problems which he regarded as important for the future development of mathematics. One of these ("Hilbert's 3rd problem") asked whether the analogue of the plane dissection theorem held in 3-space. The question comes down to the *following: suppose we have two tetrahedra with equal height and with bases of equal area. According to a well-known formula, the two have equal volume. The question is, is it possible to dissect one of the tetrahedra into a finite number of polyhedral parts, which can be reassembled to form the other? It turned out that the answer is No; this was proved by Max Dehn between the date of Hilbert's speech and its subsequent publication.*

Some of the other problems in Hilbert's list remain unsolved; many of them were indeed extremely important in the development of 20th Century mathematics.

12. A *subfield* of $\mathbb{R}$ is a (non-empty) subset which is closed under addition, subtraction, multiplication and division: that is, if you take the sum, difference, product or quotient of any two elements in the subset, (or even of the same element twice), you get another element of the subset (we don't divide by 0, of course). The set $\mathbb{Q}$ of rational numbers is an example. The aim of this exercise is to show that the set of *constructible lengths* in the plane, together with their negatives, make up a subfield of $\mathbb{R}$.

(i) For practice, show that the set of all real numbers which can be written in the form $a + b\sqrt{2}$, with a, b rational numbers, is a subfield of $\mathbb{R}$.

(ii) Show that if we start with a line segment and take its length to be our unit, then the set S of lengths we can get by using ruler and compass constructions beginning with just this segment, is closed under addition, multiplication and division. Clarification: we only allow constructions involving lines through two previously contructed points, and circles with centres which are at previously constructed points and which pass through at least one previously constructed point. A point is "constructed" if it is a point of intersection of allowable lines and or circles. For example, the procedure shown below constructs a segment AC of length 2, beginning with the segment AB of length 1. The point C is a point of intersection of an allowed line and an allowed circle.

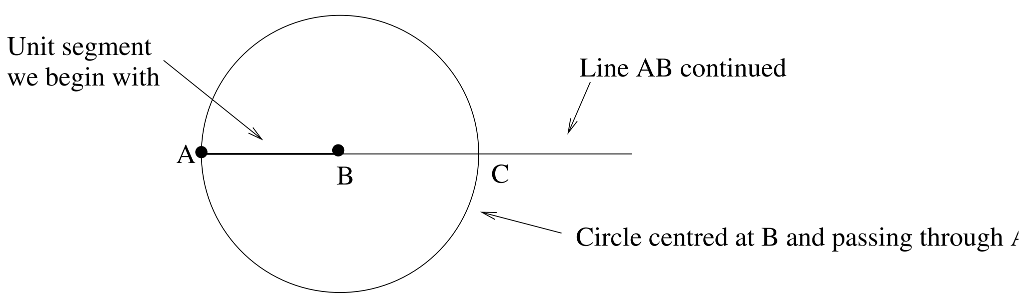


Figure 6

(ii) Show that the set $S \cup \{-x : x \in S\}$ is a subfield of $\mathbb{R}$.
(iii) Show that $S \supset \mathbb{Q}$, but that the two are not equal.

13. In this square, each vertex is joined to the midpoint of the opposite sides. Is the inner octagon regular?

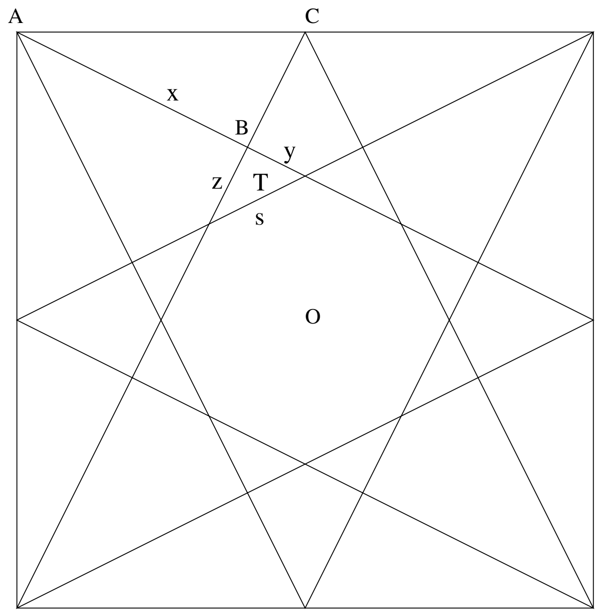


Figure 7

14. Show that in Figure 8 below

$$\frac{AS}{SB} \cdot \frac{BT}{TC} \cdot \frac{CP}{PD} \cdot \frac{DQ}{QE} \cdot \frac{ER}{RA} = 1.$$

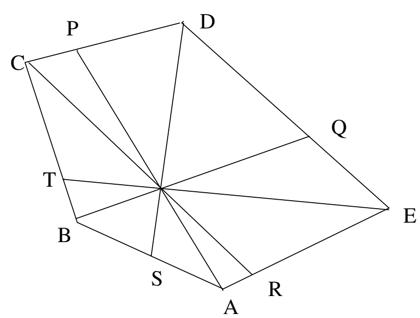


Figure 8

Is this equality a *sufficient* condition for the five lines AP, BQ, CR, DS, ET all to pass through the same point?

Can you generalise to a polygon of $2n+1$ edges (in place of the triangle in Ceva's Theorem and the pentagon here)?